

Available online at www.sciencedirect.com

SCIENCE @ DIRECT®

Topology 45 (2006) 643–671

TOPOLOGY

www.elsevier.com/locate/top

Circle actions on simply connected 5-manifolds

János Kollár*

Princeton University, Princeton, NJ 08544-1000, USA

Received 23 May 2005

Abstract

The aim of this paper is to study compact 5-manifolds which admit fixed point free circle actions. The first result implies that the torsion in the second homology and the second Stiefel–Whitney class have to satisfy strong restrictions. We then show that for simply connected 5-manifolds these restrictions are necessary and sufficient.

© 2006 Elsevier Ltd. All rights reserved.

Keywords: Circle action; 5-manifold; Seifert bundle; Sasakian geometry

It is easy to see that a simply connected compact 5-manifold L admits a free circle action iff $H_2(L, \mathbb{Z})$ is torsion free and the classification of free circle actions up to diffeomorphism is equivalent to the classification of simply connected compact 4-manifolds plus the action of their diffeomorphism group on the second cohomology (cf. [9, Proposition 10]).

Motivated by some questions that arose in connection with the study of complex analytic Seifert \mathbb{C}^* -bundles [14,16], this paper investigates compact 5-manifolds that admit circle actions where the stabilizer of every point is finite, that is, fixed point free circle actions. We show that in this case $H_2(L, \mathbb{Z})$ can contain torsion, but the torsion and the second Stiefel–Whitney class have to satisfy strong restrictions. We then show that for simply connected manifolds these restrictions are necessary and sufficient for the existence of a fixed point free circle action.

* Tel.: +1 085441000; fax: +1 6092581367.

E-mail address: kollar@math.princeton.edu.

Definition 1. Let M be any manifold. Write its second homology as a direct sum of cyclic groups of prime power order

$$H_2(M, \mathbb{Z}) = \mathbb{Z}^k + \sum_{p,i} (\mathbb{Z}/p^i)^{c(p^i)} \quad \text{for some } k = k(M), \quad c(p^i) = c(p^i, M). \quad (1.1)$$

The numbers $k, c(p^i)$ are determined by $H_2(M, \mathbb{Z})$ but the subgroups $(\mathbb{Z}/p^i)^{c(p^i)} \subset H_2(M, \mathbb{Z})$ are usually not unique. One can choose the decomposition (1.1) such that the second Stiefel–Whitney class map

$$w_2 : H_2(M, \mathbb{Z}) \rightarrow \mathbb{Z}/2$$

is zero on all but one summand $\mathbb{Z}/2^n$. This value n is unique and it is denoted by $i(M)$ [1]. This invariant can take up any value n for which $c(2^n) \neq 0$, besides 0 and ∞ . Alternatively, $i(M)$ is the smallest n such that there is an $\alpha \in H_2(M, \mathbb{Z})$ such that $w_2(\alpha) \neq 0$ and α has order 2^n .

The existence of a fixed point free differentiable circle action puts strong restrictions on H_2 and on w_2 .

Theorem 2. *Let L be a compact 5-manifold with $H_1(L, \mathbb{Z}) = 0$ which admits a fixed point free differentiable circle action. Then:*

- (1) *For every prime p , we have at most $k + 1$ nonzero $c(p^i)$ in (1.1). That is, $\#\{i : c(p^i) > 0\} \leq k + 1$.*
- (2) *One can arrange that $w_2 : H_2(L, \mathbb{Z}) \rightarrow \mathbb{Z}/2$ is the zero map on all but the $\mathbb{Z}^k + (\mathbb{Z}/2)^{c(2)}$ summands in (1.1). That is, $i(L) \in \{0, 1, \infty\}$.*
- (3) *If $i(L) = \infty$ then $\#\{i : c(2^i) > 0\} \leq k$.*

These conditions are sufficient for simply connected manifolds:

Theorem 3. *Let L be a compact, simply connected 5-manifold. Then L admits a fixed point free differentiable circle action if and only if $w_2 : H_2(L, \mathbb{Z}) \rightarrow \mathbb{Z}/2$ satisfies the conditions (2(1–3)).*

The conditions are especially transparent for spin homology spheres.

Example 4. Let $c(p^i)$ be any sequence of even natural numbers, only finitely many nonzero. By [24], there is a unique simply connected, spin, compact 5-manifold L such that $H_2(L, \mathbb{Z}) \cong \sum_{p,i} (\mathbb{Z}/p^i)^{c(p^i)}$.

By Theorem 2, this L admits a fixed point free differentiable circle action iff for every prime p , at most one of the $c(p), c(p^2), c(p^3), \dots$ is nonzero.

It should be noted that the proof does *not* give a classification of all fixed point free S^1 -actions on any 5-manifold. In fact, we exhibit infinitely many topologically distinct fixed point free S^1 -actions on every L as in Theorem 3. In principle the classification of all S^1 -actions on 5-manifolds is reduced to a question on four-dimensional orbifolds, but the four-dimensional question is rather complicated.

5. The classification of fixed point free circle actions on 3-manifolds was considered by Seifert [23]. If M is a 3-manifold with a fixed point free circle action then the quotient space $F := M/S^1$ is a surface (without boundary in the orientable case). The classification of these *Seifert fibered* 3-manifolds $f : M \rightarrow F$ is thus equivalent to the classification of fixed point free circle actions. It should be noted

that already in this classical case, it is conceptually better to view the base surface F not as a 2-manifold but as a two-dimensional *orbifold*, see [22] for a detailed survey from this point of view.

The classification of circle actions on 4-manifolds is treated in [6,7]. Here the quotient is a 3-manifold (with boundary corresponding to the fixed points) endowed with additional data involving links and certain weights.

A generalization of Seifert bundles to higher dimensions was considered in [19]. In essence, this paper considers the case when L is a real hypersurface in a complex manifold Y with a \mathbb{C}^* -action, Y/\mathbb{C}^* is a complex manifold and L is invariant under the induced S^1 -action. The computations of [19] are topological in nature, and use only that $Y/\mathbb{C}^* = L/S^1$ is a real manifold, and the fixed point set of every element of S^1 is oriented. These assumptions do not hold in general, and we follow a somewhat different approach.

Foundational questions concerning circle actions are also considered in [11].

The proofs of Theorems 2 and 3 follow the path of [23,19]. We start with any manifold L with a fixed point free circle action, and consider the quotient space $X := L/S^1$. X is usually not a manifold, only an orbifold, but we consider it with a much richer orbifold structure (X, Δ) where $\Delta = \sum (1 - (1/m_i))D_i$ is a formal sum of codimension 2 closed subspaces $D_i \subset X$. The main technical aspect of the proof is to understand how to relate invariants of L and invariants of the orbifold (X, Δ) .

In order to prove Theorem 2, we try to compute the Leray spectral sequence

$$H^i(X, R^j f_* \mathbb{Z}_L) \Rightarrow H^{i+j}(L, \mathbb{Z}).$$

This is similar to the Gysin sequence used in [19], but the Leray spectral sequence is better suited to the current situation. We end up computing $H_2(L, \mathbb{Z})$ in terms of the orbifold (X, Δ) , but the formula involves $H^{\dim X - 3}(X, \mathbb{Z})$ which I cannot control in general. If $\dim L = 5$ then this is $H^1(X, \mathbb{Z})$ and it vanishes if $H_1(L, \mathbb{Z}) = 0$.

To see that the restrictions of Theorem 2 are sufficient, we provide examples of Seifert bundles $L \rightarrow X$ with $X = (k+1)\#\mathbb{CP}^2$, a connected sum of $k+1$ copies of \mathbb{CP}^2 , and the $D_i \subset X$ are smooth surfaces intersecting transversally. It is somewhat lucky that these special cases cover all possibilities. For these examples we compute $\pi_1(L)$, $w_2(L)$ and $H_2(L, \mathbb{Z})$. Everything is easier since we do not have to worry about orbifold points of X . We then conclude the proof by using the structure theorem of simply connected compact 5-manifolds due to Smale and Barden.

Theorem 6 (Smale [24], Barden [1]). *Let L be a simply connected compact 5-manifold. Then L is uniquely determined by $H_2(L, \mathbb{Z})$ and the second Stiefel–Whitney class map $w_2 : H_2(L, \mathbb{Z}) \rightarrow \mathbb{Z}/2$.*

Furthermore, there is such a 5-manifold iff there is an integer $k \geq 0$ and a finite Abelian group A such that either

- (1) $H_2(L, \mathbb{Z}) \cong \mathbb{Z}^k + A + A$ and $w_2 : H_2(L, \mathbb{Z}) \rightarrow \mathbb{Z}/2$ is arbitrary, or
- (2) $H_2(L, \mathbb{Z}) \cong \mathbb{Z}^k + A + A + \mathbb{Z}/2$ and w_2 is projection on the $\mathbb{Z}/2$ -summand.

My original interest in this topic came from complex geometry. A method of Kobayashi [13], generalized in [2,4], allows one to construct positive Ricci curvature Einstein metrics on L from a positive Ricci curvature orbifold Kähler–Einstein metric on (X, Δ) if the base orbifold (X, Δ) has a complex structure. The existence of a positive Ricci curvature orbifold Kähler–Einstein metric on (X, Δ) imposes strong restrictions. These were explored in [14,16]. It seemed to me, however, that behind the

conditions coming from complex geometry, there were some weaker but non obvious topological restrictions as well.

In fact, the $k = 0$ case of (2(1)) first appeared in [16, Corollary 81] as a restriction on Seifert bundles over algebraic orbifolds. Now we see that this restriction is imposed not by the presence of an algebraic structure but by the topological circle action. On the other hand, in (49) we exhibit additional, this time non-topological, restrictions on Seifert bundles over algebraic orbifolds.

As Prof. Tuschmann explained to me, manifolds with circle actions also appear naturally in the Cheeger–Fukaya–Gromov theory of collapse. Namely, any collapse of a simply connected compact manifold with bounded curvature and diameter happens along the orbits of a torus action. See [20,21] and the references there.

1. Local classification of S^1 -actions

Definition 7. Let M be a differentiable manifold with a differentiable circle action $\sigma : S^1 \times M \rightarrow M$. I usually think of S^1 as a subgroup of \mathbb{C}^* . Pick a point $p \in M$ which is not a fixed point and let $O(p) \subset M$ be the orbit of p . The stabilizer of p , denoted by $\text{Stab}_p \subset S^1$, is cyclic of order $m = m(p)$ and we can choose a canonical generator $e^{2\pi i/m} \in \text{Stab}_p$.

Let H_p be a codimension 1 submanifold containing p , transversal to $O(p)$ and invariant under Stab_p . Let T_p denote the tangent space of H_p at p with its induced faithful Stab_p -action. This action of Stab_p on T_p is the *stabilizer* or *slice* representation. A neighborhood of $O(p)$ in M is S^1 -equivariantly diffeomorphic to

$$S^1 \times T_p / \text{Stab}_p, \quad (7.1)$$

where

- (i) the S^1 -action is the natural S^1 -action on itself, and
- (ii) Stab_p acts on S^1 by multiplication as a subgroup and on T_p by the *inverse* of the stabilizer representation.

Thus the local structure of $\sigma : S^1 \times M \rightarrow M$ near any orbit is determined by the stabilizer representation of Stab_p on T_p .

If the action $\sigma : S^1 \times M \rightarrow M$ has no fixed points, set $X := M/S^1$ and let $f : M \rightarrow X$ denote the quotient map. We call $f : M \rightarrow X$ the *Seifert bundle* associated to the circle action. Later we modify this definition slightly and view X not as a topological space but as a differentiable orbifold.

Every fiber of f is an S^1 -orbit, thus a circle. For $x \in X$, the stabilizer Stab_p is independent of $p \in f^{-1}(x)$. Its order is called the *multiplicity* of the fiber $f^{-1}(x)$ and it is denoted by $m(x)$ or $m(x, M)$ if there is some doubt as to which M we work with.

Given $x \in X$ and $p \in f^{-1}(x)$, an open neighborhood of x is also realized as T_p / Stab_p . This gives X the structure of a *cyclic orbifold*. That is, it is patched together from orbifold charts of the form $\mathbb{R}^n / (\text{cyclic group})$ by a linear action.

We give a detailed local description of the orbifold structure on M/S^1 later.

Definition 8. We say that $\sigma : S^1 \times M \rightarrow M$ has *orientable stabilizer representations* if the representation of Stab_p on T_p is orientation preserving for every p .

If M itself is orientable, then every element of a connected group acting on M preserves orientation, hence the stabilizer representations are all orientable.

This condition is also satisfied in many other cases when M is not orientable. The nonorientable stabilizer representation case is anomalous already in dimension 3 [22].

9 (Real representations of cyclic groups). The orientation preserving irreducible real representations of a cyclic group \mathbb{Z}/m are the trivial representation $R_{m,0}$ and the two-dimensional representations

$$R_{m,j} : \mathbb{Z}/m \ni 1 \mapsto \begin{pmatrix} \cos \frac{2\pi j}{m} & \sin \frac{2\pi j}{m} \\ -\sin \frac{2\pi j}{m} & \cos \frac{2\pi j}{m} \end{pmatrix}, \quad j = 1, \dots, m-1.$$

If m is even then $R_{m,m/2}$ decomposes as the sum of two orientation reversing representations.

If V is a two-dimensional faithful representation of \mathbb{Z}/m , then either V is orientation preserving or $m = 2$.

$R_{m,j}$ is orientation reversing isomorphic to $R_{m,m-j}$. A faithful irreducible real representation is orientation reversing isomorphic to itself only for $m = 2$.

Any orientation preserving real representation of \mathbb{Z}/m can be written as the direct sum of orientation preserving irreducible real representations. (This is not quite unique as $R_{m,j} + R_{m,j}$ and $R_{m,m-j} + R_{m,m-j}$ are orientation preserving isomorphic.) Thus every orientation preserving real representation on \mathbb{R}^{2n} can be obtained from a complex representation on \mathbb{C}^n by forgetting the complex structure. This correspondence is, however, not entirely natural, as we need to specify an orientation on each irreducible subrepresentation.

From (7.1) we obtain the following:

Lemma 10. *Let M be a differentiable manifold with a circle action $\sigma : S^1 \times M \rightarrow M$ with orientable stabilizer representations. Given integers m and c_1, \dots, c_{m-1} , the set of all points $M^0(m, c_1, \dots, c_{m-1}) \subset M$ where $\text{Stab}_p = \mathbb{Z}/m$ and the representation of Stab_p on T_p is isomorphic to*

$$\sum c_j R_{m,j} + (\text{trivial representation})$$

is a smooth submanifold of codimension $2 \sum c_j$ (or empty).

11 (Codimension 2 fixed points). By (10), the codimension 2 fixed points correspond to stabilizer representations

$$R_{m,j} + (\text{trivial representation}).$$

Let us denote the corresponding subset of M by $M^0(m, j)$. Note that for now we have a some nonuniqueness since we cannot distinguish $M^0(m, j)$ from $M^0(m, m-j)$. This will be rectified later by fixing some orientations. (This notation gives two possible meanings to $M^0(2, 1)$, but we end up with the same submanifold.)

Fix a point $p \in M^0(m, j)$. Depending on the orientation of T_p , the stabilizer representation is $R_{m,j}$ or $R_{m,m-j}$. If $m \geq 3$, then these are not orientation preserving isomorphic, so fixing say $R_{m,j}$ gives a well defined orientation of T_p .

As we move the point p in a connected component of $M^0(m, j)$, we get an orientation of T_p for every p . Thus we obtain:

Lemma 12. *Notation and assumptions as in (10). If $m \geq 3$, the normal bundle of $M^0(m, j)$ in M is orientable.*

Note. If $\dim M = 3$ then each connected component of $M^0(m, j)$ is a single S^1 -orbit, hence naturally oriented. In the cases considered by [19], the normal bundle to $M^0(m, j)$ has a complex structure hence a natural orientation. It seems to me that in general there is no natural choice, and these orientations have to be chosen by hand.

If T_p has dimension n and the corresponding stabilizer representation is $R_{m,j} + (\text{trivial representation})$, then we can write

$$T_p \cong \mathbb{C}_z + \mathbb{R}^{n-2},$$

and the representation is given by multiplication by ε^j on \mathbb{C}_z where $\varepsilon = e^{2\pi i/m}$ and $(j, m) = 1$. Thus we can write

$$T_p/\text{Stab}_p \cong \mathbb{C}_x + \mathbb{R}^{n-2} \quad \text{where } x = z^m.$$

In particular, at these points the quotient M/S^1 is a manifold.

Definition 13. Let M be a differentiable manifold with a circle action $\sigma : S^1 \times M \rightarrow M$ with orientable stabilizer representations. Let $M^s \subset M$ be the closed set of points $p \in M$ where the invariant subspace of the stabilizer representation has codimension at least 4.

Set $X = M/S^1$ with quotient map $f : M \rightarrow X$ and $X^s = M^s/S^1$. As noted above, $X^0 := X \setminus X^s$ is a manifold and

$$f \left(\bigcup_{mj} M^0(m, j) \right) \subset X^0$$

is a closed submanifold of codimension 2. Let its connected components be D_i^0 . We see in (15(4)) that their closures $D_i \subset X$ are suborbifolds.

Each D_i^0 lies in the image of a unique $M^0(m, j)$. This assigns a natural number $m = m_i$ to D_i . We introduce the formal notation

$$\left(X, \sum_i \left(1 - \frac{1}{m_i} \right) D_i \right)$$

to denote the *base orbifold* of $f : M \rightarrow X$. We call

$$f : M \rightarrow \left(X, \sum_i \left(1 - \frac{1}{m_i} \right) D_i \right)$$

the *Seifert bundle* associated to the circle action $\sigma : S^1 \times M \rightarrow M$. Sometimes we use the shorthand $\Delta = \sum_i (1 - (1/m_i)) D_i$.

(The choice of the coefficients $1 - (1/m_i)$ comes from complex geometry where the orbifold first Chern class is given by the formula $c_1(X) - \sum_i (1 - (1/m_i))[D_i]$.)

We see in (14) that the data (X, D_i, m_i) determine the orbifold structure of X .

From (12) we conclude that the normal bundle of $D_i^0 \subset X^0$ is orientable if $m_i \geq 3$.

If $(X, \sum_i (1 - (1/m_i))D_i)$ is an orbifold with oriented normal bundles $N_{D_i, X}$ for $m_i \geq 3$ then this gives an orientation to the normal bundle of $M^0(m, j)$ for $m \geq 3$ and this distinguishes $M^0(m, j)$ from $M^0(m, m - j)$. Thus each D_i^0 with $m_i \geq 3$ is in the image of a unique $M^0(m_i, j_i)$. Since the stabilizer representation is faithful, j_i is relatively prime to m_i , so $b_i j_i \equiv 1 \pmod{m_i}$ has a unique solution $1 \leq b_i < m_i$. If $m_i = 2$ then $j_i = 1$ so we can take $b_i = 1$.

The pair (m_i, b_i) is called the *orbit invariant* along D_i . (It is denoted by (α_i, β_i) in [19].) Again I emphasize that while m_i and the unordered pair $\{b_i, m_i - b_i\}$ are determined by $f : M \rightarrow X$, one needs an orientation of the normal bundle of D_i^0 to determine b_i itself.

We see in (15(6)) that the data

$$\left(X, \sum_i \frac{b_i}{m_i} D_i \right)$$

determine M locally on X if X is smooth. If X is not a manifold, one needs further local invariants at the singular points of X .

On the other hand, $(X, \sum_i (b_i/m_i)D_i)$ does not determine M globally, but the different choices are obtained by “twisting” with $H^2(X, \mathbb{Z})$ (32).

14 (Codimension 4 fixed points). As we noted in (9), an n -dimensional orientable real representation of \mathbb{Z}/m can be written as $\mathbb{C}^k + \mathbb{R}^{n-2k}$ where the action is trivial on \mathbb{R}^{n-2k} and on \mathbb{C}^k it is given by

$$(z_1, \dots, z_k) \mapsto (\varepsilon^{j_1} z_1, \dots, \varepsilon^{j_k} z_k) \quad \text{where } \varepsilon = e^{2\pi i/m}.$$

The summand \mathbb{R}^{n-2k} does not give any interesting contribution, and we concentrate on the \mathbb{C}^k part. As a shorthand, we denote the corresponding quotient by

$$\mathbb{C}^k \Big/ \frac{1}{m} (j_1, \dots, j_k). \quad (14.1)$$

We write $\mathbb{C}_{\mathbf{z}}^k$ or $\mathbb{C}_{z_1, \dots, z_k}^k$ to indicate the name of the coordinates.

The corresponding Seifert bundle is given locally by

$$(S_v^1 \times \mathbb{C}_{z_1, \dots, z_k}^k) \Big/ \frac{1}{m} (1, -j_1, \dots, -j_k) + \mathbb{R}^{n-2k} \rightarrow \mathbb{C}_{z_1, \dots, z_k}^k \Big/ \frac{1}{m} (j_1, \dots, j_k) + \mathbb{R}^{n-2k},$$

where the sign change is coming from (7.1(ii)). We usually drop the uninteresting \mathbb{R}^{n-2k} in the sequel.

It is also useful to extend S_v^1 to \mathbb{C}_v^1 and consider the quotient

$$(\mathbb{C}_v \times \mathbb{C}_{z_1, \dots, z_k}^k) \Big/ \frac{1}{m} (1, -j_1, \dots, -j_k) \rightarrow \mathbb{C}_{z_1, \dots, z_k}^k \Big/ \frac{1}{m} (j_1, \dots, j_k),$$

which is the corresponding Seifert \mathbb{C}^* -bundle extended by the zero section.

Given j_1, \dots, j_k and m , set

$$c_i := \gcd(j_1, \dots, \widehat{j_i}, \dots, j_k, m), \quad d_i := j_i c_i / C \quad \text{and} \quad C := \prod c_i. \quad (14.2)$$

Note that the c_i are pairwise relatively prime and C/c_i divides j_i . Observe that $\mathbb{Z}/c_i \subset \mathbb{Z}/m$ acts trivially on all but the i th coordinate of \mathbb{C}^k , so it is a quasi reflection.

In particular, $(z_i = 0)$ is the closure of the unique connected component of $M^0(c_i, j_i)$ intersecting this chart.

The conditions

$$r \equiv j_i \pmod{c_i} \quad \text{and} \quad c_{i'} | r \quad \text{for } i' \neq i$$

imply $r \equiv j_i \pmod{C}$. Thus the codimension 2 orbit invariants (c_i, b_i) (with $b_i j_i \equiv 1 \pmod{c_i}$) determine (m, j_1, \dots, j_k) , and hence the local structure of the Seifert bundle, if $m = \prod c_i$.

The quotient of $\mathbb{C}_{\mathbf{z}}^k$ by $\mathbb{Z}/C \cong \sum \mathbb{Z}/c_i$ is again an affine space $\mathbb{C}_{\mathbf{x}}^k$ with $x_i = z_i^{c_i}$. Thus, as a topological space

$$\mathbb{C}_{\mathbf{z}}^k \bigg/ \frac{1}{m} (j_1, \dots, j_k) \cong \mathbb{C}_{\mathbf{x}}^k \bigg/ \frac{1}{m/C} (d_1, \dots, d_k).$$

The fixed point set of every nonidentity element of $\mathbb{Z}/(m/C)$ has complex codimension ≥ 2 , thus $\mathbb{C}_{\mathbf{x}}^k/(\mathbb{Z}/(m/C))(d_1, \dots, d_k)$ is a manifold only if $m/C = 1$.

We can summarize these results as follows.

Proposition 15. *Let M be a differentiable manifold with a circle action $\sigma : S^1 \times M \rightarrow M$ with orientable stabilizer representations. Let $p \in M^s$ be a point with stabilizer representation*

$$R_{m, j_1} + \dots + R_{m, j_k} + \mathbb{R}^{n-2k} \cong \mathbb{C}_{\mathbf{z}}^k + \mathbb{R}^{n-2k}.$$

Let c_i be defined as in (14.2). Then

- (1) *An open dense subset of $(z_i = 0)$ is contained in $M^0(c_i, j_i)$. These are the only $M^0(m, j)$ whose closure contains p .*
- (2) *The c_i are pairwise relatively prime and $\prod c_i$ divides m .*
- (3) *M/S^1 is a manifold at the image of p iff $m = \prod c_i$. In this case*

$$\mathbb{C}_{\mathbf{z}}^k \bigg/ \frac{1}{m} (j_1, \dots, j_k) \cong \mathbb{C}_{\mathbf{x}}^k \quad \text{with } x_i = z_i^{c_i}.$$

Translating these into global terms we get the following:

- (4) *The closures $M(m, j)$ of $M^0(m, j) \subset M$ are smooth and intersect each other transversally.*
- (5) *If $M(m_1, j_1) \cap M(m_2, j_2) \neq \emptyset$ then $(m_1, m_2) = 1$.*
- (6) *M/S^1 is a manifold iff for every $p \in M$*

$$|\text{Stab}_p| = \prod_{M(m, j) \ni p} m.$$

In this case the pairs $\{(m, j) : M(m, j) \ni p\}$ determine the S^1 -action in a neighborhood of the orbit $O(p)$.

Some special properties of the five-dimensional case are worth emphasizing:

16. Let L be a differentiable 5-manifold with a circle action $\sigma : S^1 \times L \rightarrow L$ with orientable stabilizer representations and $f : L \rightarrow (X, \sum (1 - (1/m_i))D_i)$ the corresponding Seifert bundle. Then

- (1) $X^s \subset X$ is finite and $X \setminus X^s$ is a manifold,
- (2) each $D_i \subset X$ is a 2-manifold and at most 2 of them pass through any point of X .

2. The cohomology groups of Seifert bundles

In working with group actions on manifolds and taking various quotients, one frequently runs into orbifolds. It is therefore convenient to define the notion of Seifert bundles in a rather general setting. In order to avoid pointless complications, let us assume from now on that every topological space is a CW complex.

Definition 17. A *generalized Seifert bundle* over X is a topological space M together with an S^1 -action and a continuous map $f : M \rightarrow X$ such that X has an open covering $X = \bigcup_i U_i$ such that for every i the preimage $f : f^{-1}(U_i) \rightarrow U_i$ is fiber preserving S^1 -equivariantly homeomorphic to a “standard local generalized Seifert bundle”

$$f_i : (S^1 \times V_i)/(\mathbb{Z}/m_i) \rightarrow U_i.$$

Here V_i is a topological space with a \mathbb{Z}/m_i -action such that $V_i/(\mathbb{Z}/m_i) \cong U_i$ and the \mathbb{Z}/m_i -action on $S^1 \times V_i$ is the diagonal action given on S^1 by a homomorphism $\phi_i : \mathbb{Z}/m_i \rightarrow S^1$ composed with the action of S^1 on itself. The action of S^1 on itself gives the S^1 action on Y .

In order to avoid nontrivial orbifold structures on M , we always assume that the \mathbb{Z}/m_i -action on $S^1 \times V_i$ is fixed point free outside a codimension 2 set.

For $x \in X$ let $U_i \ni x$ be an open subset as above. Let $v \in V_i$ be a preimage of x . If $\text{Stab}_v \subsetneq \mathbb{Z}/m_i$ is a proper subgroup, then there are open subsets $x \in U_x \subset U_i$ and $v \in V_x \subset V_i$ such that $U_x = V_x/\text{Stab}_v$ and we can also describe our generalized Seifert bundle locally as

$$f^{-1}(U_x) \cong (S^1 \times V_x)/\text{Stab}_v.$$

The order of the group Stab_v depends only on x , and it is called the *multiplicity* of the fiber of $f : M \rightarrow X$ over x . It is denoted by $m(x)$ or $m(x, M)$.

18 (Maps between generalized Seifert bundles). Let $f : M \rightarrow X$ be a generalized Seifert bundle. The S^1 -equivariant homeomorphisms $h : M \rightarrow M$ such that $f \circ h = f$ are the multiplications $p \mapsto \phi(f(p)) \cdot p$ where $\phi : X \rightarrow S^1$ is any continuous function, cf. [11, 3.1].

It is more interesting to look at higher degree maps $h : M_1 \rightarrow M_2$ between generalized Seifert bundles.

Let $f : M \rightarrow X$ be a generalized Seifert bundle and $\mathbb{Z}/m \subset S^1$ a finite subgroup. Then the S^1 action descends to an $S^1/(\mathbb{Z}/m)$ -action on $M/(\mathbb{Z}/m)$ and $f/(\mathbb{Z}/m) : M/(\mathbb{Z}/m) \rightarrow X$ is another generalized Seifert bundle.

Even if M is manifold, in general $M/(\mathbb{Z}/m)$ is only an orbifold.

The case when $m = m(X)$ is the least common multiple of the multiplicities $m(x)$ is especially useful. We denote this quotient by $f/\mu : M/\mu \rightarrow X$. Since every stabilizer $\text{Stab}_p : p \in M$ is contained in $\mathbb{Z}/m(X)$, we conclude that $f/\mu : M/\mu \rightarrow X$ is a locally trivial S^1 -bundle. Locally trivial S^1 -bundles are classified by their Chern class $c_1((M/\mu)/X) \in H^2(X, \mathbb{Z})$ and we define the *Chern class* of the generalized Seifert bundle $f : M \rightarrow X$ as

$$c_1(M) = c_1(M/X) := \frac{1}{m(X)} c_1((M/\mu)/X) \in H^2(X, \mathbb{Q}).$$

Usually it is not an integral cohomology class.

Our aim is to obtain information about the integral cohomology groups of a generalized Seifert bundle $f : M \rightarrow (X, \Delta)$ in terms of (X, Δ) and the Chern class of M/X .

The cohomology groups $H^i(M, \mathbb{Z})$ are computed by a Leray spectral sequence whose E_2 term is

$$E_2^{i,j} = H^i(X, R^j f_* \mathbb{Z}_M) \Rightarrow H^{i+j}(M, \mathbb{Z}).$$

Every fiber of f is S^1 , so $R^j f_* \mathbb{Z}_M = 0$ for $j \geq 2$ and the only interesting higher direct image is $R^1 f_* \mathbb{Z}_M$. Our first task is to compute this sheaf and its cohomology groups. Next we consider the edge homomorphisms in the spectral sequence

$$\delta_i : H^i(X, R^1 f_* \mathbb{Z}_M) \rightarrow H^{i+2}(X, \mathbb{Z})$$

and identify them, at least modulo torsion, with cup product with the Chern class $c_1(M/X)$.

In some cases of interest, these data completely determine the cohomology groups, and even the topology, of M . Some of these instances are discussed in [19,14,16].

Proposition 19. *Let $f : M \rightarrow X$ be a generalized Seifert bundle.*

- (1) *There is a natural isomorphism $\tau_M : R^1 f_* \mathbb{Q}_M \cong \mathbb{Q}_X$.*
- (2) *There is a natural injection $\tau_M : R^1 f_* \mathbb{Z}_M \hookrightarrow \mathbb{Z}_X$ which is an isomorphism over points where $m(x) = 1$.*
- (3) *If $U \subset X$ is connected then*

$$\tau_M(H^0(U, R^1 f_* \mathbb{Z}_M)) = m(U) \cdot H^0(U, \mathbb{Z}) \cong m(U) \cdot \mathbb{Z},$$

where $m(U)$ is the lcm of the multiplicities of all fibers over U .

Proof. Pick $x \in X$ and a small contractible neighborhood $x \in V \subset X$. Then $f^{-1}(V)$ retracts to $S^1 \subset f^{-1}(x)$ and (together with the orientation of S^1) this gives a distinguished generator $\rho \in H^1(f^{-1}(V), \mathbb{Z})$. This in turn determines a cohomology class $(1/m(x))\rho \in H^1(f^{-1}(V), \mathbb{Q})$. These normalized cohomology classes are compatible with each other and give a global section of $R^1 f_* \mathbb{Q}_M$. Thus $R^1 f_* \mathbb{Q}_M = \mathbb{Q}_X$ and we also obtain the injection $\tau : R^1 f_* \mathbb{Z}_M \hookrightarrow \mathbb{Z}_X$ as in (2).

If $U \subset X$ is connected, a section $b \in \mathbb{Z} \cong H^0(U, \mathbb{Z}_U)$ is in $\tau(R^1 f_* \mathbb{Z}_M)$ iff $m(x)$ divides b for every $x \in U$. This is exactly (3). \square

20. Given $f : M \rightarrow X$, consider the quotient map $\pi : M \rightarrow M/\mu = M/(\mathbb{Z}/m(X))$ defined in (18). Apply (19(1)) to f and f/μ to get isomorphisms

$$\mathbb{Q}_X \xrightarrow{\tau_M^{-1}} R^1(f/\mu)_* \mathbb{Q}_{M/\mu} \xrightarrow{\pi^*} R^1 f_* \mathbb{Q}_M \xrightarrow{\tau_{M/\mu}} \mathbb{Q}_X,$$

whose composite is multiplication by $m(X)$.

Thus the map π^* induces isomorphisms between the spectral sequences

$$H^i(X, R^j f_* \mathbb{Q}_{M/\mu}) \Rightarrow H^{i+j}(M/\mu, \mathbb{Q}) \quad \text{and} \quad H^i(X, R^j f_* \mathbb{Q}_M) \Rightarrow H^{i+j}(M, \mathbb{Q}),$$

where $\pi^* : H^i(X, R^j f_* \mathbb{Q}_{M/\mu}) \rightarrow H^i(X, R^j f_* \mathbb{Q}_M)$ should be thought of as multiplication by $m(X)$.

Since $f/\mu : M/\mu \rightarrow X$ is a locally trivial circle bundle, the edge homomorphisms

$$H^i(X, R^1(f/\mu)_* \mathbb{Q}_{M/\mu}) \rightarrow H^{i+2}(X, \mathbb{Q})$$

are cup product with $c_1((M/\mu)/X)$. Since $c_1(M/X) = (1/m(X))c_1((M/\mu)/X)$ we see that the edge homomorphisms

$$\delta_i : H^i(X, R^1 f_* \mathbb{Q}_M) \rightarrow H^{i+2}(X, \mathbb{Q})$$

are cup product with $c_1(M/X)$.

Furthermore,

$$\tau(\pi^* H^0(X, R^1(f/\mu)_* \mathbb{Z}_{M/\mu})) = m(X) \cdot H^0(X, \mathbb{Z})$$

and so it agrees with $\tau(H^0(X, R^1 f_* \mathbb{Z}_M))$.

Thus we obtain the following:

Corollary 21. *Notation as above. The quotient map $\pi : M \rightarrow M/\mu$ (defined in (18)) induces an isomorphism*

$$H^0(X, R^1 f_* \mathbb{Z}_M) \cong \pi^* H^0(X, R^1(f/\mu)_* \mathbb{Z}_{M/\mu}).$$

Modulo torsion, the edge homomorphisms

$$\delta_i : H^i(X, R^1 f_* \mathbb{Z}_M) \rightarrow H^{i+2}(X, \mathbb{Z})$$

are identified with cup product with the Chern class $c_1(M/X)$. If X is connected, the image of

$$\delta : H^0(X, R^1 f_* \mathbb{Z}_M) \rightarrow H^2(X, \mathbb{Z})$$

is generated by $c_1(M/\mu) = m(X)c_1(M/X)$.

It is not clear to me how to describe the edge homomorphisms on the torsion. Since $c_1(M/X)$ is not an integral class, I do not even have a plausible guess.

Looking at the beginning of the Leray spectral sequence, we obtain:

Corollary 22. *Notation and assumptions as above.*

- (1) $H^1(M, \mathbb{Q}) = 0$ iff $H^1(X, \mathbb{Q}) = 0$ and $c_1(M/X) \neq 0$.
- (2) If $H^1(M, \mathbb{Q}) = 0$ then $\dim H^2(M, \mathbb{Q}) = \dim H^2(X, \mathbb{Q}) - 1$.

We see that (19) describes the sheaf $R^1 f_* \mathbb{Z}_M$ completely in terms of (X, Δ) , but it is not always easy to compute its cohomologies based on this description. There are, however, some cases where this is quite straightforward.

Proposition 23. *Let M be a differentiable manifold with a circle action $\sigma : S^1 \times M \rightarrow M$ with orientable stabilizer representations and $f : M \rightarrow (X, \sum (1 - (1/m_i))D_i)$ the corresponding Seifert bundle. Set*

$$K := \ker \left[\mathbb{Z}_X \rightarrow \sum_i \mathbb{Z}_{D_i}/m_i \right].$$

Then

- (1) there is an injection $\tau : R^1 f_* \mathbb{Z}_M \hookrightarrow K$ with quotient sheaf Q ,
- (2) $\text{Supp } Q$ is the set of non-manifold points of X , and
- (3) $\dim \text{Supp } Q \leq \dim X - 4$.

Proof. Pick a point $x \in X$ and let $m(x)$ denote the multiplicity of the Seifert fiber over x . Pick a small neighborhood $x \in V_x$. Then $H^0(V_x, R^1 f_* \mathbb{Z}_M) = m(x)\mathbb{Z}$ by (19).

Let $C(x)$ be the product of those m_i for which $x \in D_i$ and note that by (15), m_i and m_j are relatively prime if $D_i \cap D_j \neq \emptyset$. Thus $H^0(V_x, K) = C(x)\mathbb{Z}$. By (15), $C(x)$ divides $m(x)$ and X is a manifold at x iff $m(x) = C(x)$. \square

This allows us to compute some of the cohomology groups of $R^1 f_* \mathbb{Z}_M$.

24. From (23(3)) we conclude that $H^i(X, R^1 f_* \mathbb{Z}_M) = H^i(X, K)$ for $i \geq \dim X - 2$ and we have a long exact sequence computing $H^i(X, K)$. Thus we get information on the three top cohomology groups $H^i(X, R^1 f_* \mathbb{Z}_M)$. Set $\dim X = d$.

The top cohomology is the easiest:

$$H^d(X, R^1 f_* \mathbb{Z}_M) \cong H^d(X, \mathbb{Z}). \quad (24.1)$$

For the next one, we have an exact sequence

$$H^{d-2}(X, \mathbb{Z}) \rightarrow \sum_i H^{d-2}(D_i, \mathbb{Z}/m_i) \rightarrow H^{d-1}(X, R^1 f_* \mathbb{Z}_M) \rightarrow H^{d-1}(X, \mathbb{Z}).$$

If M is orientable, then the D_i are orientable if $m_i \geq 3$, thus $H^{d-2}(D_i, \mathbb{Z}/m_i) \cong \mathbb{Z}/m_i$ for every i . Thus we obtain:

24.2 Claim. *If M is orientable and $H^{d-1}(X, \mathbb{Z}) = 0$ then*

$$\begin{aligned} H^{d-1}(X, R^1 f_* \mathbb{Z}_M) &= \text{coker} \left[H^{d-2}(X, \mathbb{Z}) \rightarrow \sum_i H^{d-2}(D_i, \mathbb{Z}/m_i) \right] \\ &= \text{coker} \left[H^{d-2}(X, \mathbb{Z}) \rightarrow \sum_i \mathbb{Z}/m_i \right]. \end{aligned}$$

The last relevant piece of the long exact sequence is

$$H^{d-3}(X, \mathbb{Z}) \rightarrow \sum_i H^{d-3}(D_i, \mathbb{Z}/m_i) \rightarrow H^{d-2}(X, R^1 f_* \mathbb{Z}_M) \rightarrow H^{d-2}(X, \mathbb{Z}).$$

We are especially interested in the torsion in $H^{d-2}(X, R^1 f_* \mathbb{Z}_M)$. Most of it is coming from $\sum_i H^{d-3}(D_i, \mathbb{Z}/m_i)$, but $H^{d-3}(X, \mathbb{Z})$ and the torsion in $H^{d-2}(X, \mathbb{Z})$ influence it. In general these are hard to control, but we get the following:

24.3 Claim. *Assume that M is orientable, $H^{d-3}(X, \mathbb{Z}) = 0$ and $H^{d-2}(X, \mathbb{Z})$ is torsion free. Then*

$$H_{tors}^{d-2}(X, R^1 f_* \mathbb{Z}_M) \cong \sum_i H^{d-3}(D_i, \mathbb{Z}/m_i).$$

25 (Proof of (2(1))). Let L be a compact 5-manifold with a fixed point free circle action with orientable stabilizer representations. (By (12), the latter is automatic if $H_1(L, \mathbb{Z}) = 0$.) Let

$$f: L \rightarrow \left(X, \sum \left(1 - \frac{1}{m_i} \right) D_i \right)$$

be the corresponding Seifert bundle. By (16), X has finitely many non-manifold points X^s with complement $X^0 = X \setminus X^s$.

Any abelian cover of X^0 gives an abelian cover of $L \setminus L^s$, which then extends to an abelian cover of L . Thus we conclude that $H_1(L, \mathbb{Z}) = 0$ implies that $H_1(X^0, \mathbb{Z}) = 0$. By Lefschetz duality the latter gives that $H^3(X, \mathbb{Z}) = H^3(X, X^s, \mathbb{Z}) \cong H_1(X^0, \mathbb{Z}) = 0$. Furthermore, the torsion in $H^2(X, \mathbb{Z})$ is isomorphic to the torsion in $H_1(X, \mathbb{Z})$ hence again zero.

Let $b_1(D_i)$ denote $\dim H_1(D_i, \mathbb{Z}/2)$. Thus $H_1(D_i, \mathbb{Z}/m_i) = (\mathbb{Z}/m_i)^{b_1(D_i)}$ since D_i is orientable whenever $m_i \geq 3$.

Thus if $H_1(X^0, \mathbb{Z}) = 0$ then the E^2 -term of the Leray spectral sequence

$$H^i(X, R^j f_* \mathbb{Z}_L) \Rightarrow H^{i+j}(L, \mathbb{Z})$$

has the form

$$\begin{array}{ccccc} \mathbb{Z} & (\text{torsion}) & \mathbb{Z}^{k+1} + \sum_i (\mathbb{Z}/m_i)^{b_1(D_i)} & H^3(X, R^1 f_* \mathbb{Z}_L) & \mathbb{Z}, \\ \mathbb{Z} & 0 & \mathbb{Z}^{k+1} & 0 & \mathbb{Z}. \end{array}$$

One can read off the cohomology of L from this spectral sequence. Let us start with a criterion for the vanishing of $H_1(L, \mathbb{Z})$.

Proposition 26. *Let L be a compact, orientable 5-manifold with a Seifert bundle structure $f: L \rightarrow (X, \sum (1 - (1/m_i))D_i)$ such that $H^3(X, \mathbb{Z}) = 0$.*

(1) *There is a surjection*

$$H_1(L, \mathbb{Z}) \twoheadrightarrow H^3(X, R^1 f_* \mathbb{Z}_L) = \text{coker} \left[H^2(X, \mathbb{Z}) \rightarrow \sum_i H^2(D_i, \mathbb{Z}/m_i) \right].$$

(2) *Assume that $H^3(X, R^1 f_* \mathbb{Z}_L) = 0$ and X is smooth. Then the order of $H_1(L, \mathbb{Z})$ is d where d is defined by the condition*

$$c_1(L/\mu) = d \cdot (\text{primitive cohomology class}) \in H^2(X, \mathbb{Z}),$$

(where primitive := not a nontrivial multiple of any cohomology class.)

(3) *Thus if X is smooth then $H_1(L, \mathbb{Z}) = 0$ iff*

(a) $H^2(X, \mathbb{Z}) \rightarrow \sum_i H^2(D_i, \mathbb{Z}/m_i)$ is surjective, and

(b) $c_1(L/\mu) \in H^2(X, \mathbb{Z})$ is primitive.

Proof. By duality, $H_1(L, \mathbb{Z}) \cong H^4(L, \mathbb{Z})$. The spectral sequence shows that $H^4(L, \mathbb{Z}) \twoheadrightarrow H^3(X, R^1 f_* \mathbb{Z}_L)$, hence using (24.2) we obtain the first claim.

By (21), $c_1(L/\mu) \in H^2(X, \mathbb{Z})$ generates the image of the differential

$$\delta_0: \mathbb{Z} \cong H^0(X, R^1 f_* \mathbb{Z}_L) \rightarrow H^2(X, \mathbb{Z}).$$

Thus if $c_1(L/\mu) = d \cdot \beta$ is not a primitive element, then $E_{0,2}^3$ contains d -torsion, which survives in $H^2(L, \mathbb{Z})$, hence the order of $H_1(L, \mathbb{Z})$ is at least d .

Assume that X is smooth and write $c_1(L/\mu) = d \cdot \beta$ where β is primitive. Since cup product is a perfect pairing on $H^2(X, \mathbb{Z})$, there is an $\alpha \in H^2(X, \mathbb{Z})$ such that $m(X)c_1(L) \cup \alpha = d$. Thus $c_1(L) \cup m(X)\alpha = d$. Since $m_i | m(X)$ for every i ,

$$m(X)H^2(X, \mathbb{Z}) \subset \ker \left[H^2(X, \mathbb{Z}) \rightarrow \sum_i H^2(D_i, \mathbb{Z}/m_i) \right]$$

and this kernel is the \mathbb{Z}^{k+1} summand of $H^2(X, R^1 f_* \mathbb{Z}_L)$. Hence $d = c_1(L) \cup m(X)\alpha$ is in the image of

$$\delta_2: H^2(X, R^1 f_* \mathbb{Z}_L) \xrightarrow{c_1(L/X) \cup} H^4(X, \mathbb{Z}).$$

Thus the order of $H_1(L, \mathbb{Z})$ also divides d . \square

Corollary 27. *Let L be a compact 5-manifold with $H_1(L, \mathbb{Z}) = 0$, $\dim H_2(L, \mathbb{Q}) = k$ and a Seifert bundle structure $f: L \rightarrow (X, \sum (1 - (1/m_i))D_i)$. Then $\#\{i: p|m_i\} \leq k + 1$.*

Proof. Looking at (26(1)) modulo p , we obtain surjections

$$\mathbb{Z}^{k+1} \cong H^2(X, \mathbb{Z}) \twoheadrightarrow \sum_i H^2(D_i, \mathbb{Z}/m_i) \twoheadrightarrow \sum_{i: p|m_i} \mathbb{Z}/p.$$

Thus $\#\{i: p|m_i\} \leq k + 1$. \square

From the spectral sequence we also get that the torsion in $H^3(L, \mathbb{Z})$ is isomorphic to the torsion in $H^3(X, R^1 f_* \mathbb{Z}_M)$ which in turn is computed in (24.3). Thus we obtain:

Proposition 28. *Let L be a compact 5-manifold with $H_1(L, \mathbb{Z}) = 0$ and a Seifert bundle structure $f : L \rightarrow (X, \sum (1 - (1/m_i))D_i)$. Then there are isomorphisms*

$$\sum_i (\mathbb{Z}/m_i)^{b_1(D_i)} \cong \sum_i H^1(D_i, \mathbb{Z}/m_i) \cong H_{tors}^3(L, \mathbb{Z}).$$

Dually, we can construct a basis of $H_{2,tors}(L, \mathbb{Z})$ as follows:

Choose loops $\gamma_{ij} \subset D_i$, giving a basis of $H_1(D_i, \mathbb{Z}/m_i)$. Then $\Gamma_{ij} := f^{-1}(\gamma_{ij}) \subset L$ is a 2-cycle which is m_i -torsion and

$$H_{2,tors}(L, \mathbb{Z}) = \sum_{ij} (\mathbb{Z}/m_i)[\Gamma_{ij}].$$

If p^{a_i} is the largest p power dividing m_i then the p part of $H_2(L, \mathbb{Z})$ is

$$\sum_i (\mathbb{Z}/p^{a_i})^{b_1(D_i)},$$

and by (27) there are at most $k + 1$ summands. This proves (2(1)). \square

3. Construction of Seifert bundles

Definition 29. Let X be a manifold and $D \subset X$ a codimension 2 closed submanifold with a tubular neighborhood $D \subset U \subset X$ and oriented normal bundle N_D . Thus we can view N_D as a complex line bundle over D . Choose an identification $j : U \cong N_D$. Composing with the bundle map $N_D \rightarrow D$ gives a retraction $\pi : U \rightarrow D$. Thus $\pi^* N_D$ is a complex line bundle over U and it has a section $s_D : u \rightarrow (u, j(u))$ which is nowhere zero on $U \setminus D$. We can thus glue $\pi^* N_D$ with the trivial complex line bundle on $X \setminus D$ to get a complex line bundle $\mathcal{O}_X(D)$ with a section that vanishes along D .

If $D = \sum c_i D_i$ is a formal integral linear combination of codimension 2 closed submanifolds D_i with oriented normal bundles N_{D_i} , then we define

$$\mathcal{O}_X(D) := \bigotimes_i \mathcal{O}_X(D_i)^{\otimes c_i}.$$

If $c_i \geq 0$ then $\mathcal{O}_X(D)$ has a natural section which vanishes along D_i with multiplicity c_i .

One can also define the cohomology class of D by $[D] := c_1(\mathcal{O}_X(D)) \in H^2(X, \mathbb{Z})$. The map $D \mapsto [D]$ is linear in the c_i , hence it extends to rational linear combinations giving $[D] \in H^2(X, \mathbb{Q})$.

Theorem 30. *Let X be a manifold and $D_i \subset X$ codimension 2 closed submanifolds with oriented normal bundles. Let $1 \leq b_i < m_i$ be integers and B a complex line bundle on X . Assume that*

- (1) $(b_i, m_i) = 1$ for every i ,
- (2) $(m_i, m_j) = 1$ if $D_i \cap D_j \neq \emptyset$, and

- (3) the D_i intersect transversally. Then:
- (4) There is a Seifert bundle $f : M = M(B, \sum (b_i/m_i)D_i) \rightarrow X$ such that
- (a) it has orbit invariants D_i, m_i, b_i , and
 - (b) $c_1(M/X) = c_1(B) + \sum (b_i/m_i)[D_i]$.
- (5) Every Seifert bundle with orbit invariants D_i, m_i, b_i is of the form $M(B, \sum (b_i/m_i)D_i) \rightarrow X$ for some B .
- (6) The set of all such Seifert bundles forms a principal homogeneous space under $H^2(X, \mathbb{Z})$ where the action corresponds to changing B .
- (7) The properties (4(a–b)) uniquely determine M iff $H^2(X, \mathbb{Z})$ is torsion free.

Proof. Write $D = \sum (b_i/m_i)D_i$ and choose $m > 0$ such that every m_i divides m . In order to construct $M(B, \sum (b_i/m_i)D_i) \rightarrow X$ start with the rank 2 complex vector bundle

$$h : E := \mathcal{O}_X(mD) \otimes B^{\otimes m} + \mathcal{O}_X\left(\sum D_i\right) \otimes B \rightarrow X.$$

Since $m \sum D_i \geq mD$, the natural section of $\mathcal{O}_X(m \sum D_i - mD)$ gives a map

$$\sigma : \mathcal{O}_X(mD) \otimes B^{\otimes m} \rightarrow \mathcal{O}_X\left(m \sum D_i\right) \otimes B^{\otimes m}.$$

Define an auxiliary topological space $N \subset E$ to be the set of all points

$$\{(t, u, x) : h(u) = h(t) = x \text{ and } u^m = \sigma(t)\}.$$

We see that N is usually not normal, but we write down its normalization $\bar{N} \rightarrow N$ explicitly, and we show that $\bar{N} \setminus (\text{zero section})$ is a Seifert \mathbb{C}^* -bundle whose unit circle bundle is $M(B, \sum (b_i/m_i)D_i)$.

The key point is to get the local structure of N .

For $x \in X$, one can choose an open neighborhood in the form $x \in U_x \cong \mathbb{C}^k + \mathbb{R}^{n-2k}$ with complex coordinates x_1, \dots, x_k such that $(x_i=0)$ are the components of $\sum D_i$ that pass through x . After reindexing the D_i , we can assume that $D_i = (x_i=0)$ with orbit invariants (m_i, b_i) .

Over this chart, we can write $\sigma(t) = t \prod x_i^{m(1-b_i/m_i)}$ thus N is locally defined by the equation

$$u^m = t \prod x_i^{m(1-b_i/m_i)}.$$

Set $m_x = m_1 \cdots m_k$, $m = m_x m'$ and define the c_i by the condition

$$\sum c_i b_i \frac{m_x}{m_i} \equiv -1 \pmod{m_x}.$$

This is solvable since $\gcd(m_x/m_1, \dots, m_x/m_k) = 1$. In the notation of (14),

$$\begin{aligned} M_x &:= (\mathbb{C}_v + \mathbb{C}_{z_1, \dots, z_k}^k) \Big/ \frac{1}{m_x} \left(1; -c_1 \frac{m_x}{m_1}, \dots, -c_k \frac{m_x}{m_k} \right) + \mathbb{R}^{n-2k} \\ f_x \downarrow \\ U_x &:= \mathbb{C}_{z_1, \dots, z_k}^k \Big/ \frac{1}{m_x} \left(c_1 \frac{m_x}{m_1}, \dots, c_k \frac{m_x}{m_k} \right) + \mathbb{R}^{n-2k} \end{aligned}$$

is a Seifert \mathbb{C}^* -bundle extended by the zero section. Observe that the functions

$$s = v^{m_x}, \quad x_i := z_i^{m_i} \quad \text{and} \quad u := v \prod z_i^{m_i - b_i}$$

are invariant under the \mathbb{Z}/m_x -action. Set $t := s^{m'}$. Then

$$u^m = v^m \prod z_i^{m(m_i - b_i)} = t \prod x_i^{m(1 - b_i/m_i)},$$

hence we obtain a map $M_x \rightarrow N \cap h^{-1}(U_x)$ which gives the normalization. Thus the M_x patch together to $M \rightarrow N$ which is a Seifert \mathbb{C}^* -bundle extended by the zero section. Furthermore, composing with the first projection $pr_1 : E \rightarrow \mathcal{O}_X(mD) \otimes B^{\otimes m}$ we get $\pi : M \rightarrow \mathcal{O}_X(mD) \otimes B^{\otimes m}$ which is the quotient of M by \mathbb{Z}/m . Thus the first Chern class of M/X is

$$c_1(M/X) = \frac{1}{m}(c_1(mD) + c_1(B^{\otimes m})) = c_1(D) + c_1(B).$$

The rest follows from (32). \square

Corollary 31. *Notation as in (30). Set $E_i := f^{-1}(D_i)$. The normal bundle of $E_i \subset M$ is orientable consistently with the normal bundle of D_i and $f^*c_1(B) + \sum_i b_i[E_i] = 0$.*

Proof. Consider the projection maps $p : M \rightarrow \mathcal{O}_X(\sum D_i) \otimes B$ and $h_2 : \mathcal{O}_X(\sum D_i) \otimes B \rightarrow X$. The pull back $h_2^*\mathcal{O}_X(\sum D_i) \otimes B$ has a tautological section U which vanishes only along the zero section. In the local charts used in the proof of (30) this section is denoted by u . From the formula $u = v \prod z_i^{m_i - b_i}$ we see that p^*U vanishes along E_i with multiplicity $m_i - b_i$. Thus

$$c_1\left(f^*\mathcal{O}_X\left(\sum D_i\right) \otimes B\right) = \sum (m_i - b_i)[E_i].$$

Since $f^*[D_i] = m_i[E_i]$, this becomes $f^*c_1(B) = -\sum b_i[E_i]$. \square

Let X be a topological space. Continuous sections of $S^1 \times X \rightarrow X$ form a sheaf, denoted by S_X^1 . Its cohomology groups are denoted by $H^i(X, S^1)$.

Let C_X^0 denote the sheaf of continuous functions. This sheaf is soft and so it has no higher cohomologies (cf. [5, II.9]). Thus the long exact cohomology sequence of

$$0 \rightarrow \mathbb{Z}_X \rightarrow C_X^0 \rightarrow S_X^1 \rightarrow 0$$

shows that $H^i(X, S^1) \cong H^{i+1}(X, \mathbb{Z})$ for $i \geq 1$.

The following rather standard result, closely related to [11, 4.5], provides an approach to the global description of Seifert bundles.

Proposition 32. *Let X be a topological space and $X = \bigcup U_i$ an open cover. Assume that over each U_i we have a Seifert bundle $Y_i \rightarrow U_i$ and there are S^1 -equivariant homeomorphisms $\phi_{ij} : Y_j|_{U_{ij}} \cong Y_i|_{U_{ij}}$.*

- (1) *There is an obstruction element in the torsion subgroup $H_{tors}^2(X, S^1) \cong H_{tors}^3(X, \mathbb{Z})$ such that there is a global Seifert bundle $Y \rightarrow X$ compatible with these local structures iff the obstruction element is zero.*

- (2) The set of all such global Seifert bundles, up to S^1 -equivariant homeomorphisms, is either empty or forms a principal homogeneous space under $H^1(X, S^1) \cong H^2(X, \mathbb{Z})$.
- (3) The action of $H^2(X, \mathbb{Z})$ on the Chern classes is addition.

Proof. By (18), the isomorphisms ϕ_{ij} can be changed only to $\alpha_{ij}\phi_{ij}$ for any $\alpha_{ij} \in H^0(U_{ij}, S^1)$. These patchings define a global Seifert bundle iff

$$\alpha_{ik}\phi_{ik} = \alpha_{ij}\phi_{ij}\alpha_{jk}\phi_{jk} \quad \text{for every } i, j, k.$$

This is equivalent to

$$\alpha_{ij}\alpha_{jk}\alpha_{ki} = (\phi_{ij}\phi_{jk}\phi_{ki})^{-1} \quad \text{for every } i, j, k. \quad (32.3)$$

The products $(\phi_{ij}\phi_{jk}\phi_{ki})^{-1} \in H^0(U_{ijk}, S^1)$ satisfy the cocycle condition, and they define an element of $H^2(X, S^1)$, called the obstruction. One can find $\{\alpha_{ij}\}$ satisfying (32.3) iff the obstruction is zero.

Replacing the Y_j by Y_j/μ as in (18) changes the isomorphisms over $U_i \cap U_j$ to ϕ_{ij}^M , hence the obstruction corresponding to the Seifert bundles Y_j/μ is the M th power of the original obstruction.

The quotients Y_j/μ are all S^1 -bundles, and these can always be globalized to the trivial S^1 -bundle. Thus the obstruction is torsion.

Two choices $\{\alpha_{ij}\}$ and $\{\alpha'_{ij}\}$ give isomorphic Seifert bundles iff there are isomorphisms $\delta_i : Y_i \cong Y_i$ (viewed as elements of $H^0(U_i, S^1)$) such that

$$\alpha'_{ij}\alpha_{ij}^{-1} = \delta_i\delta_j^{-1}|_{U_{ij}}.$$

Thus $\{\alpha'_{ij}\alpha_{ij}^{-1}\}$ corresponds to a class in $H^1(X, S^1)$ and we also get (3). \square

By (30) the obstruction vanishes if X is a manifold and the D_i are orientable, but there are even complex orbifold examples where the obstruction is nonzero [15, Example 35].

4. The second Stiefel–Whitney class

We start the computation of the second Stiefel–Whitney class of a Seifert bundle by two key examples.

Example 33. Given $1 \leq b < m$ such that $(m, b) = 1$ consider the map

$$f : S^1_s \times \mathbb{C}_z \times \mathbb{C}_y^* \mapsto \mathbb{C}_x \times \mathbb{C}_y^* \quad \text{given by } (s, z, y) \mapsto (s^b z^m, y),$$

It has a Seifert bundle structure where the S^1_t -action is given by

$$(t) \times (s, z, y) \mapsto (t^{-m}s, t^b z, y).$$

For any y_0 , we have $f^{-1}(0, y_0) = (*, 0, y_0)$ and $\text{Stab}_{(*, 0, y_0)} = \mathbb{Z}/m$. A transverse slice at $(s_0, 0, y_0)$ is given by $(s_0, *, y_0)$ and the stabilizer representation is $z \mapsto \varepsilon^b z$ where $\varepsilon = e^{2\pi i/m}$.

Note that the tangent bundle of $S^1_s \times \mathbb{C}_z \times \mathbb{C}_y^*$ is parallelizable, hence its Stiefel–Whitney classes are all zero.

Since $H^2(\mathbb{C}_x \times \mathbb{C}_y^*, \mathbb{Z}) = 0$, we conclude from (32) that the above examples exhaust all possible Seifert bundles over the orbifold $(\mathbb{C}_x \times \mathbb{C}_y^*, (1 - (1/m))D)$, where $D = \{0\} \times \mathbb{C}_y^*$.

Example 34. For $(m, b) = (2, 1)$ there is also a version where the branch divisor is not orientable. Indeed, consider the orientation preserving involutions

$$(s, z, y) \mapsto (s, \bar{s}\bar{z}, -1/\bar{y}) \quad \text{and} \quad (x, y) \mapsto (\bar{x}, -1/\bar{y}).$$

These commute with the map f and the S^1 -action in (33). Thus we get a Seifert bundle structure on the quotient.

Since $H^2((\mathbb{C}_x \times \mathbb{C}_y^*)/(\mathbb{Z}/2), \mathbb{Z}) = 0$, as above we see that this is the only Seifert bundle over the orbifold $((\mathbb{C}_x \times \mathbb{C}_y^*)/(\mathbb{Z}/2), (1 - \frac{1}{2})D)$, where $D = (\{0\} \times \mathbb{C}_y^*/(\mathbb{Z}/2))$ is not orientable.

It is easy to see that the second Stiefel–Whitney class is nonzero on the 2-cycle $(z = 0, |y| = 1)/(\mathbb{Z}/2)$.

The next result proves (2(2)) and also gives more information about the invariant $i(L)$ defined in Definition (1).

Proposition 35. *Let L be a 5-manifold with $H_1(L, \mathbb{Z}) = 0$ having a Seifert bundle structure $f: L \rightarrow (X, \sum (1 - (1/m_i))D_i)$. Then*

- (1) $i(L) \in \{0, 1, \infty\}$, and
- (2) $i(L) = 1$ iff at least one of the D_i is nonorientable.

Proof. Fix a D_i and choose a loop $\gamma \subset D_i^0$. By (28), $\Gamma := f^{-1}(\gamma) \subset L$ is a 2-cycle which is m_i -torsion in $H_2(L, \mathbb{Z})$ and these cycles generate the torsion subgroup of $H_2(L, \mathbb{Z})$.

Let $\gamma \subset V \subset X^0$ be a tubular neighborhood. If D_i is orientable along γ , then the pair $(V, D_i \cap V)$ is diffeomorphic to $(\mathbb{C}_x \times \mathbb{C}_y^*, \{0\} \times \mathbb{C}_y^*)$, thus the restriction of $f: L \rightarrow X$ to V is diffeomorphic to one of the Seifert bundles enumerated in (33). Therefore $w_2(L) \cap [\Gamma] = 0$. Since $m_i \geq 3$ implies that D_i is orientable (12), we get the first claim.

If every D_i is orientable then we get that w_2 is zero on all the torsion, hence $i(L) \in \{0, \infty\}$. Conversely, if D_i is nonorientable along γ , then the pair $(V, D_i \cap V)$ is diffeomorphic to the one in (34), hence $w_2(L) \cap [\Gamma] \neq 0$ and so $i(L) = 1$. \square

We also need the following formula for the second Stiefel–Whitney class of a Seifert bundle. It is the topological version of the formula for the first Chern class for holomorphic Seifert bundles given in [8] and [15, Corollary 41].

Lemma 36. *Let M be a manifold with a fixed point free circle action with orientable stabilizer representations and $f: M \rightarrow (X, \sum (1 - (1/m_i))D_i)$ the corresponding Seifert bundle, X smooth. Set $E_i := f^{-1}(D_i)$. Then*

$$w_2(M) = f^*w_2(X) + \sum_i (m_i - 1)[E_i].$$

Proof. We factor f as the composite of $\pi: M \rightarrow M/\mu$ and of the projection $f/\mu: M/\mu \rightarrow X$. Since $M/\mu \rightarrow X$ is a circle bundle, $T_{M/\mu} = (f/\mu)^*T_X + (\text{trivial bundle})$, thus $w_i(M/\mu) = f^*w_i(X)$ for every i . Note that $\pi: M \rightarrow M/\mu$ is a branched covering which ramifies along the subspaces E_i and the ramification order is m_i . Thus we need to show that $w_2(M) = f^*w_2(M/\mu) + \sum_i (m_i - 1)[E_i]$. For complex manifolds and for c_1 instead of w_2 this is the Hurwitz formula.

For $\dim M \geq 5$, we can represent any homology class $H_2(M, \mathbb{Z}/2)$ by an embedded surface $S \hookrightarrow M$. We may also assume that S is transversal to $\bigcup E_i$ and near each point $E_i \cap S$ the induced map $\pi: S \rightarrow \pi(S)$ is a branched cover of degree m_i . In a neighborhood of $\pi(S)$ we can write $T_{M/\mu} = E + N$ where N is a trivial bundle, $\text{rank } E = 2$ and near each point $\pi(E_i \cap S)$ the subbundles $T_{\pi(S)} \subset T_{M/\mu}$ and $E \subset T_{M/\mu}$ agree. Thus $\pi^*w_2(M/\mu) \cap [S] = w_2(\pi^*E) \cap S$. Correspondingly one can write $T_M|_S = E' + \pi^*N$ where E' is a rank 2 bundle with induced tangent map $E' \subset \pi^*E$ whose quotient is supported at the points $E_i \cap S$ and has length $m_i - 1$ there. \square

This formula becomes easier to use if we combine it with (31) which says that $f^*c_1(B) + \sum b_i [E_i] = 0$. Thus

$$\begin{aligned} w_2(M) &= f^*w_2(X) + \sum_i (m_i - 1)[E_i] + f^*c_1(B) + \sum b_i [E_i] \\ &= f^*w_2(X) + \sum_i (m_i - 1 + b_i)[E_i] + f^*c_1(B). \end{aligned}$$

If m_i is even then b_i is odd so $m_i - 1 + b_i$ is even. If m_i is odd then $m_i - 1$ is even, so we can rewrite this as

$$w_2(M) = f^*w_2(X) + \sum_{i: m_i \text{ odd}} b_i [E_i] + f^*c_1(B).$$

Note that in integral cohomology, $m_i[E_i] = f^*[D_i]$, hence, in $H^2(M, \mathbb{Z}/2)$, we get that $[E_i] = f^*[D_i]$ if m_i is odd and $f^*[D_i] = 0$ if m_i is even. Thus we can rewrite our formula as follows.

Corollary 37. *Let $M = M(B, \sum (b_i/m_i)D_i) \rightarrow X$ be a Seifert bundle as in (30), X smooth and X, D_i orientable. Then*

$$w_2(M) = f^* \left(w_2(X) + \sum_i b_i [D_i] + c_1(B) \right).$$

38 (Proof of (2(3))). Let $f: L \rightarrow (X, \sum (1 - (1/m_i))D_i)$, be a Seifert bundle such that $H_1(L, \mathbb{Z}) = 0$. If $i(L) \neq 1$ then every D_i is orientable by (35).

By (16), there is a finite set $X^s \subset X$ such that $X^0 := X \setminus X^s$ is a manifold. Set $L^0 := f^{-1}(X^0)$. Since $L \setminus L^0$ has codimension 4, we see that $H^i(L, \mathbb{Z}/2) = H^i(L^0, \mathbb{Z}/2)$ for $i \leq 2$.

By (28), if $c(2^i) \neq 0$ for $k+1$ values of i then $\geq k+1$ of the m_j are even. Let these be D_0, \dots, D_k . Since $H^2(X, \mathbb{Z}) \rightarrow \sum_i H^2(D_i, \mathbb{Z}/m_i)$ is surjective, we conclude that

$$H^2(X, \mathbb{Z}/2) \rightarrow \sum_{i=0}^k H^2(D_i, \mathbb{Z}/2) \quad \text{is surjective.}$$

The two sides have the same rank, so we have an isomorphism. This implies that D_0, \dots, D_k form a basis of $H_2(X, \mathbb{Z}/2) = H_2(X, X^s, \mathbb{Z}/2)$.

By Lefschetz duality, $\mathcal{O}_{X^0}(D_0), \dots, \mathcal{O}_{X^0}(D_k)$ (or rather their Chern classes) form a basis of $H^2(X^0, \mathbb{Z}/2)$. Since $f^*\mathcal{O}_{X^0}(D_j) = \mathcal{O}_{L^0}(m_j E_j)$, and m_0, \dots, m_k are even, we conclude that the pull back map $f^*: H^2(X^0, \mathbb{Z}/2) \rightarrow H^2(L^0, \mathbb{Z}/2)$ is zero.

On the other hand, (37) applies and $w_2(L^0)$ is the pull back of a cohomology class from X^0 . Thus $w_2(L) = w_2(L^0) = 0$ and so $i(L) = 0$. \square

5. Seifert bundles over \mathbb{CP}^2

In this section we construct examples of Seifert bundles $f : L \rightarrow (X, \Delta)$. The base (X, Δ) is constructed as a connected sum of pairs $(\mathbb{CP}^2, \sum (1 - (1/m_i))D_i)$, where the attaching does not involve the D_i . Once the base (X, Δ) is chosen, we can vary the complex line bundle B to obtain various values of the invariant $i(L)$.

We already have a good understanding of the cohomology of L , the key additional step is to control the fundamental group as well. This is achieved by the following simple lemma.

Lemma 39. *Let $f : L \rightarrow (X, \Delta)$ be a Seifert bundle, X a manifold. If $\pi_1(X \setminus \Delta)$ is solvable then so is $\pi_1(L)$. If this holds then $\pi_1(L) = 1$ iff $H_1(L, \mathbb{Z}) = 0$.*

Proof. Since $f : L \setminus f^{-1}(\Delta) \rightarrow X \setminus \Delta$ is a circle bundle, there is an exact sequence

$$\pi_1(S^1) \rightarrow \pi_1(L \setminus f^{-1}(\Delta)) \rightarrow \pi_1(X \setminus \Delta) \rightarrow 1.$$

Since $f^{-1}(\Delta) \subset L$ has codimension 2, there is a surjection $\pi_1(L \setminus f^{-1}(\Delta)) \rightarrow \pi_1(L)$. \square

Remark 40. Although we do not need it, it is worthwhile to note that there is an exact sequence for $\pi_1(L)$ itself.

Let $(X, \sum_i (1 - (1/m_i))D_i)$ be an orbifold and $X^0 \subset X$ the smooth locus of X . The *orbifold fundamental group* $\pi_1^{orb}(X, \Delta)$ is the fundamental group of $X^0 \setminus \text{Supp } \Delta$ modulo the relations: if γ is any small loop around D_i then $\gamma^{m_i} = 1$ [25].

Note that $\pi_1^{orb}(X, \emptyset)$ may be different from $\pi_1(X)$ if X is not a manifold.

The abelianization of $\pi_1^{orb}(X^0, \Delta)$, denoted by $H_1^{orb}(X^0, \Delta)$, is called the *abelian orbifold fundamental group*. (The higher orbifold homotopy and homology groups are defined in [10].)

A straightforward generalization of the computation of the fundamental group of three-dimensional Seifert bundles (see [23, 11, 5.7]) gives the exact sequence

$$\pi_1(S^1) \rightarrow \pi_1(L) \rightarrow \pi_1^{orb}(X, \Delta) \rightarrow 1.$$

The next lemma gives a large collection of pairs $(\mathbb{CP}^2, \sum D_i)$ to work with.

Lemma 41. *Let D_1, \dots, D_s be compact surfaces. Then there are embeddings $D_i \subset \mathbb{CP}^2$ such that*

- (1) *the D_i intersect transversally,*
- (2) *if D_i is orientable, then its homology class $[D_i]$ is a generator of $H_2(\mathbb{CP}^2, \mathbb{Z})$,*
- (3) *if D_i is nonorientable, then $[D_i]$ is a generator of $H_2(\mathbb{CP}^2, \mathbb{Z}/2)$,*
- (4) *$\pi_1(\mathbb{CP}^2 \setminus (D_1 \cup \dots \cup D_s))$ is abelian.*

Proof. Let us start with \mathbb{C}^2 and for each surface with $b_1(D_i)$ even pick a complex line L_i^0 in general position and for each surface with $b_1(D_i)$ odd pick a non complex real affine 2-plane $L_i^0 \subset \mathbb{C}^2$ in general

position. Correspondingly, in \mathbb{CP}^2 we get embedded copies L_1, \dots, L_s of \mathbb{CP}^1 and of \mathbb{RP}^2 which intersect transversally and satisfy the conditions (41.2–3).

A classical lemma [26, p. 317] states (in the case of complex lines) that $\pi_1(\mathbb{CP}^2 \setminus (L_1^0 \cup \dots \cup L_s^0))$ is abelian, but the proof applies to real 2-planes in \mathbb{R}^4 as above as long as all intersections are transverse and any two of the planes do intersect. This implies that $\pi_1(\mathbb{CP}^2 \setminus (L_1 \cup \dots \cup L_s))$ is abelian.

Next we aim to attach handles to the L_i without changing the fundamental group of the complement. The key part is the following local computation.

42 (Attaching handles). Take \mathbb{R}^4 with coordinates (x, y, z, t) . A pl-embedded copy of \mathbb{R}^2 is given by the union of the two half planes $H_1 := (x \leq 0, y, 0, 0)$ and $H_2 := (0, y, 0, t > 0)$. Make 2 holes in H_1 and attach a handle $S^1 \times [-1, 1]$ inside $\mathbb{R}^3 \cong (t = 0)$ to H_1 . Depending on how this is done, the resulting surface can be orientable or nonorientable. Together with H_2 , we obtain a pl-embedded surface $D \subset \mathbb{R}^4$.

We claim that $\pi_1(\mathbb{R}^4 \setminus D) \cong \mathbb{Z}$, generated by a loop around $H_2 \subset (t > 0)$.

Indeed, any loop in $\mathbb{R}^4 \setminus D$ can be made transversal to the hyperplane $(t = 0)$. Since $(t = 0) \setminus D$ is connected, we can assume that the loop intersects $(t = 0)$ only at points where $x > 0$.

Since D is disjoint from the half space $(t < 0)$, any part of the loop in this half space can be contracted and then pushed above $(t = 0)$. Thus we homotoped the loop to the upper half space $(t > 0)$ and $\pi_1((t > 0) \setminus H_2) \cong \mathbb{Z}$.

To get the final embeddings, pick points $p_i \in L_i$ not on any of the other L_j and disjoint neighborhoods $p_i \in U_i \sim \mathbb{R}^4$ such that $L_i \cap U_i \hookrightarrow U_i$ is a linearly embedded \mathbb{R}^2 . We can attach handles to the L_i to get $D_i \subset \mathbb{CP}^2$. In doing this, we have not created new intersections and the homology class of D_i is the same as the homology class of the L_i . (With the caveat that if L_i is orientable but D_i is not then we claim only a modulo 2 equality.)

Finally we need to show that $\pi_1(\mathbb{CP}^2 \setminus (D_1 \cup \dots \cup D_s))$ is abelian. We prove that it is isomorphic to $\pi_1(\mathbb{CP}^2 \setminus (L_1 \cup \dots \cup L_s))$, which we already know to be abelian.

We can compute both of these fundamental groups using van Kampen's theorem from the $\pi_1(U_i \setminus L_i)$ (resp. $\pi_1(U_i \setminus D_i)$) and

$$\mathbb{CP}^2 \setminus (U_1 \cup \dots \cup U_s) \setminus (L_1 \cup \dots \cup L_s) = \mathbb{CP}^2 \setminus (U_1 \cup \dots \cup U_s) \setminus (D_1 \cup \dots \cup D_s).$$

Thus it is enough to prove that both are computed the same way, which amounts to proving that $\pi_1(U_i \setminus L_i) = \pi_1(U_i \setminus D_i)$. This was already done in (42). \square

Note. If the D_i are all orientable, although D_i and its normal bundle both have a complex structure, these cannot be made compatible with the (almost) complex structure of \mathbb{CP}^2 . Indeed, with the usual complex structure of \mathbb{CP}^2 , $c_1(T_{\mathbb{CP}^2}|_{D_i}) = 3$ and using the above complex structures on D_i and its normal bundle gives

$$c_1(T_{\mathbb{CP}^2}|_{D_i}) = c_1(T_{D_i}) + c_1(N_{D_i}) = 2 - 2g_i + 1 = 3 - 2g_i.$$

Construction 43 (Seifert bundles with $i(L) \in \{0, \infty\}$). Assume that we have a natural number k and a sequence of natural numbers $c(p^i)$ such that

- (1) the $c(p^i)$ are all even, and
- (2) for every prime p , at most $k + 1$ of the $c(p^i)$ are nonzero.

We want to construct Seifert bundles $f: L \rightarrow (X, \sum (1 - (1/m_i))D_i)$ with $i(L) \in \{0, \infty\}$. By (35(2)) this means that every D_i is orientable.

For every prime p arrange the p^i satisfying $c(p^i) \neq 0$ in increasing order, and put 1's at the beginning to get a sequence of length $k + 1$

$$1 = p^0 = \dots = p^0 < p^{i(1)} < \dots < p^{i(h)}.$$

If $p \neq 2$ or if $c(2^i) > 0$ for some i then for $j = 0, \dots, k$ let m_{pj} be the j th element of this sequence and set $g_{pj} = \frac{1}{2}c(m_{pj})$ where we declare that $c(1) = 0$. If $c(2^i) = 0$ for every i then set $m_{2j} = 1$ for $j < k$, $m_{2k} = 2$ and $g_{2j} = 0$ for every j .

For each $j = 0, \dots, k$, pick a copy \mathbb{CP}_j^2 of \mathbb{CP}^2 and as in (41) construct oriented surfaces $D_{pj} \subset \mathbb{CP}_j^2$ of genus g_{pj} . Let X be the connected sum of the \mathbb{CP}_j^2 with $D_{pj} \subset X$. Thus $D_{pj} \cap D_{p'j'} = \emptyset$ iff $j \neq j'$.

Let $H_j \in H^2(X, \mathbb{Z})$ denote the cohomology class of a line on \mathbb{CP}_j^2 . Then H_0, \dots, H_k is a free basis of $H^2(X, \mathbb{Z})$.

The restriction map $H^2(X, \mathbb{Z}) \rightarrow \sum_p H^2(D_{pj}, \mathbb{Z}/m_{pj})$ can be written as a sum of the individual restriction maps

$$\mathbb{Z} \cong H^2(\mathbb{CP}_j^2, \mathbb{Z}) \rightarrow \sum_p H^2(D_{pj}, \mathbb{Z}/m_{pj}) \cong \sum_p \mathbb{Z}/m_{pj} \cong \bigg/ \left(\prod_p m_{pj} \right), \quad (43.1)$$

where the last isomorphism holds since m_{pj} and $m_{p'j'}$ are relatively prime for $p \neq p'$. Thus the first condition of (26(3)) is satisfied.

On \mathbb{CP}_k^2 we have surfaces D_{pk} and assigned multiplicities m_{pk} . Let $m(X) = \prod_p m_{pk}$. Since the m_{pk} are relatively prime to each other and their product is $m(X)$, we can find $1 \leq b_{pk} < m_{pk}$ and $h_k \in \mathbb{Z}$ such that $(b_{pk}, m_{pk}) = 1$ and

$$\sum_p \frac{b_{pk}}{m_{pk}} = \frac{1}{m(X)} - h_k.$$

Set $b_{pj} = 1$ for $j < k$; these values are unimportant for us. We fix these values of b_{pj} and h_k for the rest of the proof, and study how the second Stiefel–Whitney class depends on h_0, \dots, h_{k-1} .

By (30), for any h_0, \dots, h_{k-1} there is a Seifert bundle

$$L(h_0, \dots, h_{k-1}) := L \left(\mathcal{O}_X \left(\sum_{j=0}^k h_j H_j \right), \sum_{p,j} \frac{b_{pj}}{m_{pj}} D_{pj} \right) \rightarrow X \quad (43.2)$$

and $\gamma_j \in (\prod_p m_{pj})^{-1} \mathbb{Z}$ such that

$$c_1(L(h_0, \dots, h_{k-1})) = \frac{1}{m(X)} H_k + \sum_{j=0}^{k-1} \gamma_j H_j. \quad (43.3)$$

Since m_{pk} contains p with the largest exponent, we see that $m_{pj} | m(X)$ for every j and $m(X)/m_{pj}$ is even for every $j < k$. Thus

$$\begin{aligned} c_1(L(h_0, \dots, h_{k-1})/\mu) &= m(X)c_1(L(h_0, \dots, h_{k-1})) \\ &\in H_k + 2\langle H_0, \dots, H_{k-1} \rangle. \end{aligned} \quad (43.4)$$

This implies that $c_1(L(h_0, \dots, h_{k-1})/\mu)$ is a primitive vector and so the second condition of (26(3)) also holds, implying that $H_1(L(h_0, \dots, h_{k-1}), \mathbb{Z}) = 0$. From (41) and (39) we conclude that $L(h_0, \dots, h_{k-1})$ is simply connected. By (28),

$$H_2(L(h_0, \dots, h_{k-1}), \mathbb{Z}) = \mathbb{Z}^k + \sum_{p,i} (\mathbb{Z}/p^i)^{c(p^i)}.$$

So far the h_0, \dots, h_{k-1} played no visible role in the construction. Now we aim to choose these appropriately to control $w_2(L(h_0, \dots, h_{k-1}))$.

We have a pull back map $f^* : H^2(X, \mathbb{Z}/2) \rightarrow H^2(L(h_0, \dots, h_{k-1}), \mathbb{Z}/2)$ and by (37) there is an $\eta \in H^2(X, \mathbb{Z}/2)$ such that

$$w_2(L(h_0, \dots, h_{k-1})) = f^* \left(\eta + \sum_{i=0}^{k-1} h_i H_i \right). \quad (43.5)$$

By choosing h_0, \dots, h_{k-1} suitably, we can thus assume that

$$w_2(L(h_0, \dots, h_{k-1})) = f^*(cH_k) \quad \text{for some } c. \quad (43.6)$$

We know from (21) that $c_1(L(h_0, \dots, h_{k-1})/\mu)$ is in the kernel of $f^* : H^2(X, \mathbb{Z}) \rightarrow H^2(L(h_0, \dots, h_{k-1}), \mathbb{Z})$, hence its mod 2 reduction, which is H_k by (43.4), is in the kernel of $f^* : H^2(X, \mathbb{Z}/2) \rightarrow H^2(L(h_0, \dots, h_{k-1}), \mathbb{Z}/2)$. Thus with the choices of (43.6), we have a trivial second Stiefel–Whitney class.

We have completed the first part of the existence theorem (3):

Corollary 44. *Let $c(p^i)$ be even natural numbers (only finitely many nonzero) satisfying (2(1)) for some k . Then there are Seifert bundles $f : L \rightarrow ((k+1)\#\mathbb{CP}^2, \Delta)$ such that L is simply connected, $w_2(L) = 0$ (equivalently, $i(L) = 0$) and*

$$H_2(L, \mathbb{Z}) = \mathbb{Z}^k + \sum_{p,i} (\mathbb{Z}/p^i)^{c(p^i)}.$$

Getting examples with $i(L) = \infty$ is very similar. First, by changing the choice of h_0 in (43.6), we can achieve that

$$w_2(L(h_0, \dots, h_{k-1})) = f^*(H_0 + cH_k). \quad (43.7)$$

(The notation does not show, but we have to keep in mind that f^* also depends on h_0, \dots, h_{k-1} .) We know that $H_0 + cH_k$ is not in the kernel of the pull back map between integral cohomology groups. If we can show that $H_0 + cH_k$ is not in the kernel of the pull back map between $\mathbb{Z}/2$ -cohomology groups, then we get the desired examples with $i(L) = \infty$.

By assumption, at most k of the $c(2^i)$ are nonzero, thus there is an $i \geq 1$ such that every m_{pj} is odd for $j < i$ but m_{2i} is even.

Let Y be the connected sum of the first $i + 1$ copies of \mathbb{CP}^j and $g : M \rightarrow (Y, \Delta)$ the corresponding Seifert bundle. By the above parity considerations,

$$c_1(M/\mu) \in 2\langle H_0, \dots, H_{i-1} \rangle + \mathbb{Z}H_i.$$

Only one of the m_{pj} with $j \leq i$ is even, hence from the exact sequence given by (23)

$$H^0(Y, \mathbb{Z}) \rightarrow \sum_{j \leq i} H^0(D_{pj}, \mathbb{Z}/m_{pj}) \rightarrow H^1(Y, R^1g_*\mathbb{Z}_M) \rightarrow H^1(Y, \mathbb{Z}) = 0$$

we conclude that $H^1(Y, R^1g_*\mathbb{Z}_M)$ has odd order. Thus the pull back map between integral cohomology groups sits in an exact sequence

$$\mathbb{Z} \xrightarrow{c_1(M/\mu)} H^2(Y, \mathbb{Z}) \xrightarrow{g^*} H^2(M, \mathbb{Z}) \rightarrow (\text{odd order group}).$$

This implies that modulo 2 we still get an injection

$$g^* : \langle H_0, \dots, H_{i-1} \rangle \hookrightarrow H^2(M, \mathbb{Z}/2). \quad (43.8)$$

Let $y \in Y$ be a point not on any of the D_{pj} and set $Y^0 := Y \setminus \{y\}$ and $M^0 := M \setminus g^{-1}(y)$. Then $H^2(M, \mathbb{Z}/2) = H^2(M^0, \mathbb{Z}/2)$ and we can think of M^0 as an open subset of L . Thus (43.8) implies that

$$f^* : \langle H_0, \dots, H_{i-1} \rangle \hookrightarrow H^2(L, \mathbb{Z}/2) \quad \text{is an injection.} \quad (43.9)$$

Since $f^*(H_k) = 0 \in H^2(L, \mathbb{Z}/2)$ by (43.4), we conclude that $f^*(H_0 + cH_k) \in H^2(L, \mathbb{Z}/2)$ is nonzero, giving the second existence result:

Corollary 45. *Let $c(p^i)$ be even natural numbers (only finitely many nonzero) satisfying (2(1)) and (2(3)) for some $k \geq 1$. Then there are Seifert bundles $f : L \rightarrow ((k+1)\#\mathbb{CP}^2, \Delta)$ such that L is simply connected, $i(L) = \infty$ and*

$$H_2(L, \mathbb{Z}) = \mathbb{Z}^k + \sum_{p,i} (\mathbb{Z}/p^i)^{c(p^i)}.$$

Construction 46 (Seifert bundles with $i(L) = 1$). Assume that we have a natural number k and a sequence of natural numbers $c(p^i)$ such that

- (1) the $c(p^i)$ are all even except possibly $c(2) \geq 1$, and
- (2) for every prime p , at most $k + 1$ of the $c(p^i)$ are nonzero.

We want to construct a Seifert bundle $f : L \rightarrow (X, \sum (1 - (1/m_i))D_i)$ with $i(L) = 1$. By (35(2)) we can assure this by choosing one of the D_i to be nonorientable.

The construction follows (43), with two changes:

- (1) We do not have to compute $w_2(L)$, since $i(L) = 1$ is guaranteed by having a nonorientable surface D_{2j} .

- (2) There is a unique j with $m_{2j} = 2$ and then we want $D_{2j} \subset \mathbb{CP}_j^2$ to be nonorientable with $H^1(D_{2j}, \mathbb{Z}/2) \cong (\mathbb{Z}/2)^{c(2)}$. We can find such D_{2j} by (41), but the Seifert bundle over \mathbb{CP}_j^2 has to be constructed by hand since the existence result (30) works only for orientable surfaces.

Let $N \subset \mathbb{CP}^2$ be a compact nonorientable surface which is nonzero in $H_2(\mathbb{CP}^2, \mathbb{Z}/2)$. From the sequence

$$H_2(N, \mathbb{Z}/2) \xrightarrow{\cong} H_2(\mathbb{CP}^2, \mathbb{Z}/2) \rightarrow H_2(\mathbb{CP}^2, N, \mathbb{Z}/2)$$

and Lefschetz duality we conclude that $H^2(\mathbb{CP}^2, \mathbb{Z}/2) \rightarrow H^2(\mathbb{CP}^2 \setminus N, \mathbb{Z}/2)$ is the zero map, thus

$$\text{im}[H^2(\mathbb{CP}^2, \mathbb{Z}) \rightarrow H^2(\mathbb{CP}^2 \setminus N, \mathbb{Z})] \subset 2 \cdot H^2(\mathbb{CP}^2 \setminus N, \mathbb{Z}).$$

By (30), there is a Seifert bundle

$$g : M \rightarrow \left(\mathbb{CP}^2, \sum_{p \geq 3} \left(1 - \frac{1}{m_{pj}} \right) D_{pj} \right)$$

such that $c_1(M/\mu) \in H^2(\mathbb{CP}^2, \mathbb{Z})$ is the generator. Set $N = D_{2j} \subset \mathbb{CP}^2$ and consider the restriction

$$g : M \setminus g^{-1}(N) \rightarrow \left(\mathbb{CP}^2 \setminus N, \sum_{p \geq 3} \left(1 - \frac{1}{m_{pj}} \right) D_{pj} \right).$$

As noted above

$$c_1(M \setminus g^{-1}(N)/\mu) \in H^2(\mathbb{CP}^2 \setminus N, \mathbb{Z})$$

is not primitive, but twice a generator. By (26) this means that $\pi_1(M \setminus g^{-1}(N)) = \mathbb{Z}/2$ and we have a ramified double cover

$$g' : M' \rightarrow M \rightarrow \left(\mathbb{CP}^2, \left(1 - \frac{1}{2} \right) D_{2j} + \sum_{p \geq 3} \left(1 - \frac{1}{m_{pj}} \right) D_{pj} \right).$$

This gives the required Seifert bundle with a nonorientable D_i . The rest of (43) works without changes and we obtain the final existence result:

Corollary 47. *Let $c(p^i)$ be even natural numbers for $p^i \geq 3$ (only finitely many nonzero) satisfying (2.1) for some k and $c(2) \geq 1$. Then there are Seifert bundles $f : L \rightarrow ((k+1)\#\mathbb{CP}^2, \Delta)$ such that L is simply connected, $i(L) = 1$ and*

$$H_2(L, \mathbb{Z}) = \mathbb{Z}^k + \sum_{p,i} (\mathbb{Z}/p^i)^{c(p^i)}.$$

Remark 48. Note that a surface D_{ij} with genus 0 does not contribute to $H_2(L, \mathbb{Z})$. Thus if we add new m_{ij} with $g_{ij} = 0$ to our collection, we get the same total space for the Seifert bundle. The number and

genera of the D_{ij} are easy to determine from the subset of L where the action is not free. Thus for each such L we get infinitely many topologically distinct circle actions.

6. Quasi-regular contact and Sasakian structures

An interesting case of S^1 -actions arises when (L, η) is a contact manifold and the Reeb vector field gives an S^1 -action. These are called *quasi-regular* or *almost-regular* contact structures. These are distinguished by the fact that $d\eta$ descends to an (orbifold) symplectic structure on $X = L/S^1$ and the $D_i \subset X$ are symplectic suborbifolds.

I cannot say anything useful about the general contact case, but as a first step one may consider Sasakian structures (see [3] for a recent survey). For our purposes, these are Seifert bundles over algebraic orbifolds. That is, Seifert bundles $f: L \rightarrow (X, \sum (1 - (1/m_i))D_i)$ where X is a complex algebraic (possibly singular) surface and $D_i \subset X$ are complex algebraic curves.

Since symplectic 4-manifolds are close to complex algebraic surfaces in many respects, one may hope that some features of the Sasakian case continue to hold for contact manifolds as well.

Ref. [16, Cor.81] shows that not every simply connected rational homology sphere admits a Sasakian structure, but now we see that the restrictions found there are purely topological. That is, they are obstructions to the existence of a fixed point free circle action as well.

Here is a relatively simple result which shows that Sasakian structures impose additional topological restrictions beyond those that come from the circle action itself given in Theorem 2.

Lemma 49. *Let $L \rightarrow (X, \sum (1 - (1/m_i))D_i)$ be a Seifert bundle over an algebraic orbifold. Assume that $H_1(L, \mathbb{Z}) = 0$ and*

$$H_2(L, \mathbb{Z}) = \sum_{p,i} (\mathbb{Z}/p^i)^{c(p^i)}.$$

Then there is a degree 2 polynomial q with integer coefficients such that $q(\mathbb{Z})$ contains all but 10 elements of the set $\{c(p^i)\}$.

In particular, $\{c(p^i)\}$ contains at most $12 + 2\sqrt{N}$ elements of any interval of length N .

Example 50. Let the p_i be different prime numbers and $L \rightarrow (X, \sum (1 - (1/m_i))D_i)$ a Seifert bundle over an algebraic orbifold such that $H_1(L, \mathbb{Z}) = 0$ and

$$H_2(L, \mathbb{Z}) = \sum_{i=1}^k (\mathbb{Z}/p_i)^{2i}.$$

Then (49) implies that $k \leq 23$.

On the other hand, the conditions of Theorem 2 are satisfied for any k , and so there are such Seifert bundles for any k and p_i even over $X = \mathbb{CP}^2$. The surfaces D_i , however, cannot be chosen complex algebraic for $k > 23$.

These give examples of simply connected 5-manifolds which admit a fixed point free circle action yet have no Sasakian structure.

With more careful estimates and some case analysis, one should be able to reduce 23 to about 10, but it may be hard to get a sharp result.

Proof. If X is an algebraic surface with quotient singularities, the Bogomolov–Miyaoka–Yau-type inequalities of [17, 10.8, 10.14] and [12, 9.2] imply that

$$\sum_x \left(1 - \frac{1}{r_x}\right) < e(X),$$

where r_x is the order of the fundamental group of the link of a singular point $x \in X$ and $e(X)$ is the topological Euler number. In our case $e(X) = 3$ and so X has at most 5 singular points. Thus by (16) there are at most 10 curves D_i which pass through at least one singular point.

Once D_i is contained in the smooth locus of X , its genus is computed by the adjunction formula

$$2g(D_i) = (D_i \cdot (D_i + K_X)) + 2.$$

Since X has Picard number 1, we conclude that there is a degree 2 polynomial with integer coefficients $q(t)$ such that $2g(D_i) \in q(\mathbb{Z})$.

Thus we are finished by the following easy lemma:

Lemma 51. *Let $q(t) = at^2 + bt + c$ with $a > 0$. Then the set $q(\mathbb{Z})$ intersects every interval of length N in at most $2 + 2\sqrt{N/a}$ elements.*

Example 52. Let $X = \mathbb{P}^2$ and $D_i \subset X$ be a smooth curve of degree i for $i = 3, \dots, n$, hence $g(D_i) = \binom{i-1}{2}$. Choose pairwise relatively prime integers m_i . By (30) and (28) there is a simply connected Seifert bundle $L \rightarrow (\mathbb{C}\mathbb{P}^2, \sum (1 - (1/m_i^2))D_i)$ such that

$$H_2(L, \mathbb{Z}) = \sum_{i=3}^n (\mathbb{Z}/m_i)^{2\binom{i-1}{2}}.$$

This gives $\sqrt{N} - 1$ different values $c(p^i)$ in the interval $[1, N]$ for $N = n^2$.

Remark 53. A rich source of algebraic examples is given by the Brieskorn manifolds $L = L(a_1, \dots, a_n) := (\sum z_i^{a_i} = 0) \cap S^{2n-1}$. An algorithm of [18] can be used to compute $H_*(L, \mathbb{Z})$ in any given case, but the computations can be rather lengthy and it is hard to see general patterns. Extensive lists have been obtained by Boyer, Galicki and Wraith (unpublished).

Acknowledgments

I thank A. Adem, Ch. Boyer, K. Galicki, Z. Szabó and W. Tuschmann for useful conversations and e-mails. Research was partially supported by the NSF under Grant numbers DMS-0200883 and DMS-0500198.

References

- [1] D. Barden, Simply connected five-manifolds, *Ann. Math.* 82 (2) (1965) 365–385 MR 018424241 (32 #1714).
- [2] C.P. Boyer, K. Galicki, On Sasakian-Einstein geometry, *Internat. J. Math.* 11 (7) (2000) 873–909 MR 2001k:53081.

- [3] C.P. Boyer, K. Galicki, Sasakian geometry, hypersurface singularities and Einstein metrics, *Suppl. Rend. Circ. Mat. Paler.* 75 (2005) 57–87.
- [4] C.P. Boyer, K. Galicki, M. Nakamaye, On positive Sasakian geometry, *Geom. Dedicata* 101 (1) (2003) 93–102.
- [5] G.E. Bredon, *Sheaf Theory*, McGraw-Hill, New York, 1967 MR 36 #4552.
- [6] R. Fintushel, Circle actions on simply connected 4-manifolds, *Trans. Amer. Math. Soc.* 230 (1977) 147–171 MR 0458456 (56 #16659).
- [7] R. Fintushel, Classification of circle actions on 4-manifolds, *Trans. Amer. Math. Soc.* 242 (1978) 377–390 MR 496815 (81e:57036).
- [8] H. Flenner, M. Zaidenberg, Normal affine surfaces with \mathbb{C}^* -actions, *Osaka J. Math.* 40 (4) (2003) 981–1009 MR 2 020 670.
- [9] H. Geiges, Contact structures on 1-connected 5-manifolds, *Mathematika* 38 (2) (1991/1992) 303–311 MR 1147828 (93e:57042).
- [10] A. Haefliger, Groupoïdes d’holonomie et classifiants, *Astérisque* (116) (1984) 70–97, Transversal structure of foliations, Toulouse, 1982. MR 86c:57026a.
- [11] A. Haefliger, É. Salem, Actions of tori on orbifolds, *Ann. Global Anal. Geom.* 9 (1) (1991) 37–59 MR 92f:57047.
- [12] S. Keel, J. McKernan, Rational curves on quasi-projective surfaces, *Mem. Amer. Math. Soc.* 140 (669) (1999) viii+153 MR 1610249 (99m:14068).
- [13] S. Kobayashi, Topology of positively pinched Kaehler manifolds, *Tôhoku Math. J.* 15 (2) (1963) 121–139 MR 27 #4185.
- [14] J. Kollár, Einstein metrics on connected sums of $S^2 \times S^3$, arXiv:math.DG/0402141, 2004.
- [15] J. Kollár, Seifert G_m -bundles, arXiv:math.AG/0404386, 2004.
- [16] J. Kollár, Einstein metrics on 5-dimensional Seifert bundles, *J. Geom. Anal.* 15 (3) (2005) 463–495.
- [17] J. Kollár, et al., Flips and abundance for algebraic threefolds, *Société Mathématique de France, Paris, 1992. Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, UT, August 1991*, *Astérisque* 211 (1992), MR 94f:14013.
- [18] P. Orlik, On the homology of weighted homogeneous manifolds, *Proceedings of the Second Conference on Compact Transformation Groups (Univ. Massachusetts, Amherst, MA, 1971)*, Part I, *Lecture Notes in Mathematics*, vol. 298, Springer, Berlin, 1972, pp. 260–269, MR 55 #3312.
- [19] P. Orlik, P. Wagreich, Seifert n -manifolds, *Invent. Math.* 28 (1975) 137–159 MR 50 #13596.
- [20] A. Petrunin, X. Rong, W. Tuschmann, Collapsing vs. positive pinching, *Geom. Funct. Anal.* 9 (4) (1999) 699–735 MR 1719598 (2000h:53053).
- [21] A. Petrunin, W. Tuschmann, Diffeomorphism finiteness, positive pinching, and second homotopy, *Geom. Funct. Anal.* 9 (4) (1999) 736–774 MR 1719602 (2000k:53031).
- [22] P. Scott, The geometries of 3-manifolds, *Bull. London Math. Soc.* 15 (5) (1983) 401–487 MR 705527 (84m:57009).
- [23] H. Seifert, Topologie dreidimensionaler gefaserte Räume, *Acta Math.* 60 (1932) 148–238.
- [24] S. Smale, On the structure of 5-manifolds, *Ann. of Math.* 75 (2) (1962) 38–46 MR 25 #4544.
- [25] W. Thurston, *The geometry and topology of 3-manifolds*, Princeton University, Mimeographed Notes, 1978.
- [26] O. Zariski, On the problem of existence of algebraic functions of two variables possessing a given branch curve, *Amer. J. Math.* 51 (1929) 305–328.